# B.A./B.Sc. 5th Semester (Honours) Examination, 2023 (CBCS)

## Subject : Mathematics

## Course : BMH5DSE13

## (Point Set Topology)

**Time: 3 Hours** **Full Marks: 60**

*The figures in the margin indicate full marks.*
*Candidates are required to give their answers in their own words as far as practicable.*

*Notation and symbols have their usual meaning.*

**1.** Answer *any ten* questions: 2×10=20

(a) State the axiom of choice.

(b) Define product topology.

(c) Define addition and multiplication of two cardinal numbers. 1+1

(d) If $u, v$ and $w$ are cardinal numbers then prove that $u(vw) = (uv)w$.

(e) Let $(A, \leq)$ be a totally ordered set and $x \leq y,\ x, y \in A$. Prove that $A_x \subset A_y$, where $A_x$ and $A_y$ denote the initial segments determined by $x$ and $y$ respectively.

(f) Let $(X, \tau)$ be a topological space. Prove that a subset $G$ of $X$ is open if and only if it is a neighbourhood of each of its points.

(g) Let $(X, \tau), (Y, \tau')$ and $(Z, \tau'')$ be three topological spaces. Let $f: (X, \tau) \to (Y, \tau')$ and $g: (Y, \tau') \to (Z, \tau'')$ be two continuous functions. Prove that $gof: (X, \tau) \to (Z, \tau'')$ is continuous.

(h) Define a cofinite topological space.

(i) Prove that a topological space $(X, \tau)$ is connected if and only if it has no non-empty proper subset which is both open and closed.

(j) Let $A$ and $B$ be two connected subsets in a topological space $(X, \tau)$ with $A \cap B \neq \emptyset$. Prove that $A \cup B$ is connected in $(X, \tau)$.

(k) Let $Q$ be the set of rational numbers equipped with subspace topology of usual topology of $\mathbb{R}$. Examine if $Q$ is connected.

(l) Prove that each cofinite space is compact.

(m) Let $X$ be a compact space and $Y$ be a $T_2$ space and $f: X \to Y$ be continuous. Prove that $f$ is a closed map. 2

(n) Show that a circle or a line or a parabola in $\mathbb{R}^2$ is not homeomophic to a hyperbola. 2

(o) Using the definition of a compact set, prove that the open interval (0, 1) is not a compact subset of $\mathbb{R}$, the real number space equipped with usual topology. 2

**2.** Answer *any four* questions: 5×4=20

(a) Prove that a sequentially compact metric space is compact.

(b) Let $f: (X, \tau) \to (Y, \tau')$ be a function. When is $f$ called closed? Prove that f is closed map if and only if $f(\overline{A}) \supset \overline{f(A)}$ for every $A \subset X$. 1+2+2

(c) If $u, v, w$ are cardinal numbers show that $(uv)^w = u^w \cdot v^w$.

(d) Let $(A, \leq)$ be a well ordered set. Prove that

(i) $A$ is order isomorphic to no initial segment of $A$,

(ii) if $A_x \approxeq A_y$, then $x = y$. 3+2

(e) Define a path connected space. Prove that every path connected space is connected. 1+4

(f) Prove that a topological space is locally connected if and only if each component of an open set is open. 2½+2½

**3.** Answer *any two* questions: 10×2=20

(a) (i) Prove that for any two cardinal numbers $u$ and $v$, either $u \leq v$ or $v \leq u$.

(ii) If $u, v$ and $w$ are cardinal numbers, then prove that $u^v u^w = u^{v+w}$. 5+5

(b) (i) Prove that a compact subset in a metric space is closed and bounded.

(ii) When is a topological space said to be locally connected? Is every connected space locally connected? Support your answer. (2+3)+(1+4)

(c) (i) Let $(X, \tau)$ and $(Y, \tau')$ be two topological spaces and $f: X \to Y$ be a mapping. Let $\{x_n\}$ be a sequence in $X$ converging to $x$. If $f$ is continuous then show that the sequence $\{f(x_n)\}$ converges to $f(x)$ in $Y$. Does the converse hold? Support your answer.

(ii) Let $C$ be a connected subset of a topological space $(X, \tau)$. Show that $\bar{C}$ is connected. Hence or otherwise show that component in a topological space is closed. (2+4)+(3+1)

(d) (i) Define a locally compact space. Prove that a closed subset of a locally compact space is locally compact.

(ii) If $u, v, w$ are cardinal numbers, prove that $(u^v)^w = u^{vw}$. (1+4)+5